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Boundary conditions for integrable discrete chains

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Abstract
A method of searching for boundary conditions consistent with the integrability property is suggested for discrete chains. Generating functions of conserved quantities of the corresponding finite-dimensional reductions are given in terms of the L, A pairs.

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1. Introduction

In [1] a list of double discrete chains of nonrelativistic Toda type

$$q_{m+1,n} = F(q_{m,n+1}, q_{m,n}, q_{m,n-1}, q_{m-1,n}) \tag{1}$$

is given admitting the so-called duality transformation. It consists of eight equations which are not reduced to each other by point transformations. The list contains well known equations having physical significance such as double discrete versions of the Toda chain [2, 3] and the Heisenberg equation [4], and new ones as well (more detailed references can be found in [1]).

It is convenient to represent all of the chains in the following form:

$$(T_m - 1)p(x) = (T_n - 1)r(y) \tag{2}$$

where $x = q - q_{-1,0}$, $y = q - q_{0,-1}$ and abbreviation of indices is used such that $q = q_{m,n}$, $q_{ij} = q_{m+i,n+j}$. Here T_m and T_n are the shift operators on the first and second subscripts respectively so that $T_m q = q_{1,0}$ and $T_n q = q_{0,1}$. Let us reproduce the list of chains:

$$(T_m - 1)\frac{1}{x} = (T_n - 1)\frac{1}{y} \tag{3}$$

$$(T_m - 1)e^x = (T_n - 1)e^y \tag{4}$$

$$(T_m - 1)\frac{1}{e^x - 1} = (T_n - 1)\frac{1}{e^y - 1} \tag{5}$$

$$(T_m - 1)\log x = (T_n - 1)\log y \tag{6}$$

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$$(T_m - 1)\log\left(1 - \frac{1}{x}\right) = (T_n - 1)\log\left(1 - \frac{1}{y}\right) \tag{7}$$

$$(T_m - 1)\log(\mathrm{e}^x - 1) = (T_n - 1)\log(\mathrm{e}^y - 1) \tag{8}$$

$$(T_m - 1)x = (T_n - 1)\log(\mathrm{e}^y + 1) \tag{9}$$

$$(T_m - 1)\log\left(\frac{\mathrm{e}^x - \mu}{\mathrm{e}^x - 1}\right) = (T_n - 1)\left(\frac{\mathrm{e}^y - \mu}{\mathrm{e}^y - 1}\right). \tag{10}$$

The purpose of the paper is to study cut-off conditions for these chains that preserve their integrability property. By analogy to the PDE case one can refer to them as boundary conditions consistent with integrability. The discrete chains represented in the list above are less studied from this point of view than are PDEs and differential difference lattices. An effective way to search for boundary conditions was proposed by Sklyanin [5]. It is formulated in terms of the so-called R-matrix which is a solution of the classical Yang–Baxter equation. The method originally developed for differential equations was later applied to the time-discrete models [2].

We use an alternative method based on classical symmetries of the associated linear operators which was successfully applied recently to the KdV type of PDEs [7]. The main difference between this method and the one mentioned above is that in the former case it is not necessary to use the R-matrix but only the L, A pair. The method can be applied to models of dimension higher than 1 + 1.

We start with the assumption that equation (1) is represented as a consistency condition of two linear equations of the following form (L, A pair):

$$Y(m, n+1, \lambda) = L([q], \lambda)Y(m, n, \lambda) \tag{11}$$

$$Y(m+1, n, \lambda) = A([q], \lambda)Y(m, n, \lambda) \tag{12}$$

where L and A depend on a finite number of shifts of the field variable $q = q_{m,n}$ and on the parameter λ. We impose a cut-off constraint (boundary condition) at the spatial point $n = N$ such that $\forall m$

$$q_{m,N} = f(m, q_{m,N+1}, q_{m,N+2}, \ldots, q_{m,N+M}). \tag{13}$$

Under this constraint the lattice (1) is reduced from the line $-\infty < n < +\infty$ to the semi-line $n \geqslant N + 1$.

Generally, the pair of linear equations (11), (12) admits a group (G) of classical symmetries of the form $Y \to \bar{Y} = HY$, $\lambda \to \bar{\lambda} = h(\lambda)$, i.e. the equations are not changed under such changes of variables. Here the matrix-valued function $H = H(m, [q], \lambda)$ depends only on a finite number of shifts of q. Notice that in some cases the group G may contain only the trivial identical transform.

Definition. *We will call the boundary condition (13) consistent with the L, A pair if the equation*

$$Y(m+1, N, \lambda) = A([q], \lambda)|_{q_{m,N}=F} Y(m, N, \lambda) \tag{14}$$

admits at least one additional linear transformation of the form

$$\tilde{Y}(m, N, h(\lambda)) = H(m, [q], \lambda)Y(m, N, \lambda) \qquad h = h(\lambda). \tag{15}$$

In other words, equation (12) taken at the point $n = N$ after replacing the variable $q_{m,N}$ with the rhs of (13) should get an extra linear symmetry. It is important that this symmetry does not belong to the group G.

The definition offers an algorithm for looking for integrable boundary conditions for chains. We shall demonstrate that the algorithm is simple and effective.

In section 2 we list the integrable boundary conditions for chains from (3)–(10). For chains (3), (7) L, A pairs were found by Adler and for (4), (5), (8)–(10) by Suris. For chain (6) the L, A pair is constructed by ourselves. For the chain (9) our boundary conditions are the same as those found earlier in [2, 6]. It is clear that the matrix $H = H(m, \lambda)$ defining the transformation (15) may depend on a finite number of shifts of q. In our examples below we consider only the case when H does not depend on the dynamical variables. Recall that the set of dynamical variables for chains consists of variables $q_{m,n}, q_{m-1,n}$ where $n = N+1, N+2, \dots$ and m is fixed.

Finite-dimensional reductions of the chains (3)–(10) obtained by imposing boundary conditions on two points are considered in section 3. It is shown that if both conditions are chosen consistent with the L, A pair and the involution $h = h(\lambda)$ is the same for both ends then the restricted equation has a large number of integrals of motion. Illustrative examples of finite chains and their conserved quantities are also represented.

2. Integrable boundary conditions

According to our definition above the boundary condition (13) is integrable if a pair of functions $h = h(\lambda)$, $H = H(m, \lambda) = H(m, q_{m,N}, q_{m,N+1}, \dots, q_{m,N+k}, \lambda)$ exists such that for any solution $Y(m, N, \lambda)$ of the equation $Y(m+1, N, \lambda) = A(m, N, \lambda)Y(m, N, \lambda)$ the function $\tilde{Y}(m, N, h(\lambda)) = H(m, \lambda)Y(m, N, \lambda)$ is also a solution of the same equation. Thus the following equation has to be valid:

$$H(m+1, \lambda)A(m, N, \lambda) = A(m, N, h(\lambda))H(m, \lambda). \tag{16}$$

This is the main equation for defining integrable boundary conditions.

Let us start with the first equation (see (3)) of the list. It is called the discrete Heisenberg model and has the following L, A pair:

$$L = \begin{pmatrix} \lambda - \frac{q_{-1,0}}{q-q_{-1,0}} & -\frac{qq_{-1,0}}{q-q_{-1,0}} \\ \frac{1}{q-q_{-1,0}} & \lambda + \frac{q}{q-q_{-1,0}} \end{pmatrix} \qquad A = \begin{pmatrix} \lambda - \frac{q_{0,-1}}{q-q_{0,-1}} & -\frac{qq_{0,-1}}{q-q_{0,-1}} \\ \frac{1}{q-q_{0,-1}} & \lambda + \frac{q}{q-q_{0,-1}} \end{pmatrix}.$$

Proposition. *Suppose that a boundary condition of the form (13) for the discrete Heisenberg model (3) is consistent with the L, A pair and the corresponding function $H = H(m, \lambda)$ depends only on m and λ. Then it reads as*

$$q_{m,0} = \frac{cq_{m,1} + (-1)^m a}{c + (-1)^m b q_{m,1}} \tag{17}$$

where a, b, c are arbitrary constants and $a^2 + b^2 \neq 0$.

Proof. Evidently the main equation (16) gives rise to a system of four scalar equations on the elements $h_{i,j}(m, \lambda)$ of the matrix $H(m, \lambda)$ (more exactly, we denote $h_{i,j} = (H(m, \lambda))_{i,j}$ and $\bar{h}_{i,j} = (H(m+1, \lambda))_{i,j}$)

$$\bar{h}_{11}\left(\lambda - \frac{f}{q_{m,1} - f}\right) + \bar{h}_{12}\frac{1}{q_{m,1} - f} = h_{11}\left(h(\lambda) - \frac{f}{q_{m,1} - f}\right) - h_{21}\frac{q_{m,1}f}{q_{m,1} - f} \tag{18}$$

$$-\bar{h}_{11}\frac{q_{m,1}f}{q_{m,1} - f} + \bar{h}_{12}\left(\lambda + \frac{q_{m,1}}{q_{m,1} - f}\right) = h_{12}\left(h(\lambda) - \frac{f}{q_{m,1} - f}\right) - h_{22}\frac{q_{m,1}f}{q_{m,1} - f} \tag{19}$$

$$\bar{h}_{21}\left(\lambda - \frac{f}{q_{m,1} - f}\right) + \bar{h}_{22}\frac{1}{q_{m,1} - f} = h_{21}\left(h(\lambda) + \frac{q_{m,1}}{q_{m,1} - f}\right) + h_{11}\frac{1}{q_{m,1} - f} \tag{20}$$

$$\bar{h}_{22}\left(\lambda + \frac{q_{m,1}}{q_{m,1} - f}\right) - \bar{h}_{21}\frac{q_{m,1}f}{q_{m,1} - f} = h_{22}\left(h(\lambda) + \frac{q_{m,1}}{q_{m,1} - f}\right) + h_{12}\frac{1}{q_{m,1} - f}. \tag{21}$$

Here N is taken to be zero. Let us first differentiate equation (20) with respect to the variable $q_{m,1}$. This leads to the equation

$$\left((h_{21}+\bar{h}_{21})q_{m,1}+(h_{11}-\bar{h}_{22})\right)\frac{\mathrm{d}f}{\mathrm{d}q_{m,1}}=(h_{21}+\bar{h}_{21})f+(h_{11}-\bar{h}_{22}). \qquad (22)$$

Assuming that $(h_{21}+\bar{h}_{21})^2+(h_{11}-\bar{h}_{22})^2\neq 0$ and integrating (22) one finds an explicit expression for f

$$f(q_{m,1})=\frac{q_{m,1}+(h_{11}-\bar{h}_{22})c}{1-(h_{21}+\bar{h}_{21})c} \qquad \text{or} \qquad f(q_{m,1})=\frac{\bar{h}_{22}-h_{11}}{\bar{h}_{21}+h_{21}}$$

containing the constant of integration c. The last term in the right-hand side of (18) now clearly vanishes, i.e. $h_{21}=0$, because it contains a quadratic factor $q_{m,1}^2$. The same is true of the first term in (21), so that $\bar{h}_{21}=0$. Then (20) implies $h_{11}-\bar{h}_{22}=0$. But this contradicts our assumption $(h_{21}+\bar{h}_{21})^2+(h_{11}-\bar{h}_{22})^2\neq 0$. Consequently, $h_{21}=-\bar{h}_{21}$ and $h_{11}=\bar{h}_{22}$.

Returning to the equation (20) again one gets $(h(\lambda)+\lambda+1)h_{21}=0$. So one has to examine two separate subcases: (1) $h(\lambda)=-\lambda-1$ and (2) $h_{21}=0$. Begin with the first one. Recall that it is proved above that $h_{21}=-\bar{h}_{21}$ and $h_{11}=\bar{h}_{22}$. Assume that $h_{21}\neq 0$, then it follows from (18), (21) that

$$\left((h_{22}-\bar{h}_{11})q_{m,1}+(h_{12}+\bar{h}_{12})\right)\frac{\mathrm{d}f}{\mathrm{d}q_{m,1}}=(h_{22}-\bar{h}_{11})f+(h_{12}+\bar{h}_{12}). \qquad (23)$$

If $(h_{22}-\bar{h}_{11})^2+(h_{12}+\bar{h}_{12})^2\neq 0$ then evidently

$$f(q_{m,1})=\frac{q_{m,1}+(h_{12}+\bar{h}_{12})c}{1-(h_{22}-\bar{h}_{11})c} \qquad \text{or} \qquad f(q_{m,1})=\frac{h_{12}+\bar{h}_{12}}{\bar{h}_{11}-h_{22}}.$$

Substitution of this answer into (18) gives immediately $h_{21}=0$. Consequently, our assumption $(h_{22}-\bar{h}_{11})^2+(h_{12}+\bar{h}_{12})^2\neq 0$ is wrong and $h_{22}=\bar{h}_{11}$, $\bar{h}_{12}=-h_{12}$.

Further analysis of the system (18)–(21) shows that

$$f=f(m,q_{m,1})=\frac{cq_{m,1}+(-1)^m a}{c+(-1)^m bq_{m,1}}$$

where a, b, c are arbitrary constants. The constants a and b cannot vanish simultaneously, in the opposite case one gets $q_{m,0}=q_{m,1}$ and the denominator in (3) becomes zero. The matrix H takes the form

$$H(m,\lambda)=\begin{pmatrix} 1 & (-1)^m ac(2\lambda+1) \\ (-1)^m bc(2\lambda+1) & 1 \end{pmatrix} \qquad h(\lambda)=-\lambda-1.$$

The second case gives nothing new. □

The chain might have integrable boundary conditions of a special kind which cannot be written in the form of (13); these are connected by zeros of the functions $p(x)$ and $r(y)$ in (2). For instance, the formal constraint $q_{m,0}=\infty$ cuts off the chain (3) preserving the first integrals (see example 1 below). In this special case the matrix H and the involution $h=h(\lambda)$ can also be found from (18)–(21):

$$H(m,\lambda)=\begin{pmatrix} 1 & (1+1/\lambda)^m a \\ 0 & 1 \end{pmatrix} \qquad h(\lambda)=\lambda$$

where a is an arbitrary constant.

Below we give a list of boundary conditions corresponding to other equations from the list (3)–(10).

(i) Equation (4):

$$L = \begin{pmatrix} \lambda + e^{q-q_{-1,0}} & -e^{q} \\ e^{-q_{-1,0}} & -1 \end{pmatrix} \qquad A = \begin{pmatrix} \lambda + e^{q-q_{0,-1}} & -e^{q} \\ e^{-q_{0,-1}} & -1 \end{pmatrix}$$

$$e^{-q_{m,0}} = 0 \qquad h(\lambda) = \lambda \qquad H(m,\lambda) = \begin{pmatrix} 1 & (-\lambda)^m a \\ 0 & 1 \end{pmatrix}.$$

We use a, b, c, d to denote arbitrary constants.

(ii) Equation (5):

$$L = \begin{pmatrix} \lambda + \frac{e^{q}}{e^{q}-e^{q_{-1,0}}} & -\frac{e^{q+q_{-1,0}}}{e^{q}-e^{q_{-1,0}}} \\ \frac{1}{e^{q}-e^{q_{-1,0}}} & \lambda - \frac{e^{q_{-1,0}}}{e^{q}-e^{q_{-1,0}}} \end{pmatrix} \qquad A = \begin{pmatrix} \lambda + \frac{e^{q}}{e^{q}-e^{q_{0,-1}}} & -\frac{e^{q+q_{0,-1}}}{e^{q}-e^{q_{0,-1}}} \\ \frac{1}{e^{q}-e^{q_{0,-1}}} & \lambda - \frac{e^{q_{0,-1}}}{e^{q}-e^{q_{0,-1}}} \end{pmatrix}$$

$$e^{q_{m,0}} = \frac{c e^{q_{m,1}} + (-1)^m a}{c + (-1)^m b e^{q_{m,1}}} \qquad a^2 + b^2 \neq 0$$

$$h(\lambda) = -\lambda - 1 \qquad H(m,\lambda) = \begin{pmatrix} 1 & (-1)^m ac(2\lambda+1) \\ (-1)^m bc(2\lambda+1) & 1 \end{pmatrix}.$$

In the special case $e^{q_{m,0}} = \infty$ one has $h(\lambda) = \lambda$,

$$H(m,\lambda) = \begin{pmatrix} 1 & \left(\frac{\lambda}{1+\lambda}\right)^m a \\ 0 & 1 \end{pmatrix}.$$

(iii) Equation (6):

$$L = \begin{pmatrix} \lambda + q - q_{-1,0} & \lambda - \lambda(q - q_{-1,0}) \\ 1 - q + q_{-1,0} & 1 + \lambda(q - q_{-1,0}) \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda + q - q_{0,-1} & \lambda - \lambda(q - q_{0,-1}) \\ 1 - q + q_{0,-1} & 1 + \lambda(q - q_{0,-1}) \end{pmatrix}$$

$$q_{m,0} = q_{m,1} + c \qquad h(\lambda) = \lambda$$

$$H(m,\lambda) = \begin{pmatrix} 1 + \frac{(1+c)(\lambda-c)}{c(1+\lambda)^2} g(m-1) & g(m) \\ \frac{1}{c}\left(\frac{1+c}{1+\lambda}\right)^2 g(m-1) & 1 - \frac{(1+c)(\lambda-c)}{c(1+\lambda)^2} g(m-1) \end{pmatrix}$$

where $c_1 = -\frac{1}{c}\left(\frac{\lambda-c}{1+\lambda}\right)^2$, $g(m) = \frac{\lambda(1-\lambda)}{c(1+\lambda)^2}\left(\frac{1-c_1^{m-1}}{1-c_1} + c_1^m\right)$.

(iv) Equation (7):

$$L = \begin{pmatrix} \lambda + \frac{q}{q-q_{-1,0}} & -\frac{q+\lambda}{q-q_{-1,0}} \\ \frac{q}{q-q_{-1,0}} & \lambda - \frac{q+\lambda}{q-q_{-1,0}} \end{pmatrix} \qquad A = \begin{pmatrix} \lambda + \frac{q}{q-q_{0,-1}} & -\frac{q+\lambda}{q-q_{0,-1}} \\ \frac{q}{q-q_{0,-1}} & \lambda - \frac{q+\lambda}{q-q_{0,-1}} \end{pmatrix}$$

$$q_{m,0} = q_{m,1} + c \qquad h(\lambda) = -\lambda$$

$$H(m,\lambda) = \begin{pmatrix} (-1)^{m+1}\frac{1-c_1^m}{1-c_1} & (-1)^m\left((c+1)c_1^{m-1} + 2\frac{1-c_1^{m-1}}{1-c_1}\right) \\ (-1)^m c c_1^m & (-1)^m c_1^m \end{pmatrix}$$

where $c_1 = \frac{1}{c} - 1$ and

$$q_{m,0} = c q_{m,1} \qquad h(\lambda) = -\lambda \qquad H(m,\lambda) = \begin{pmatrix} (-1)^m & 2(-1)^{m+1} \\ 0 & (-1)^{m+1} \end{pmatrix}.$$

In the special case $q_{m,0} = \infty$ the function $h = h(\lambda)$ and the matrix $H(m,\lambda)$ can be defined as follows:

$$h(\lambda) = \lambda \qquad H(m,\lambda) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(v) Equation (8):

$$L = \begin{pmatrix} \lambda + e^{q-q_{-1,0}} & -e^{q} \\ (\lambda+1)e^{-q_{-1,0}} & -1 \end{pmatrix} \qquad A = \begin{pmatrix} \lambda + e^{q-q_{0,-1}} & -e^{q} \\ (\lambda+1)e^{-q_{0,-1}} & -1 \end{pmatrix}$$

$$e^{-q_{m,0}} = 0 \qquad h(\lambda) = \lambda \qquad H(m,\lambda) = \begin{pmatrix} 1 & (-\lambda)^m a \\ 0 & 1 \end{pmatrix}.$$

(vi) Equation (9):

$$L = \begin{pmatrix} \lambda + e^{q-q_{-1,0}} & e^{q} \\ \lambda e^{-q_{-1,0}} & 0 \end{pmatrix} \qquad A = \begin{pmatrix} \lambda & e^{q} \\ \lambda e^{-q_{0,-1}} & -1 \end{pmatrix}$$

$$e^{-q_{m,0}} = be^{q_{m,1}} + a \qquad h(\lambda) = 1/\lambda$$

$$H(m,\lambda) = \begin{pmatrix} (-1/\lambda)^{m-1}\frac{1}{\lambda+1}a & (-1/\lambda)^m \\ (-1/\lambda)^m b & (-1/\lambda)^m \frac{1}{\lambda+1}a \end{pmatrix}.$$

In the special case $e^{-q_{m,0}} = 0$ we have

$$h(\lambda) = \lambda \qquad H(m,\lambda) = \begin{pmatrix} 1 & (-\lambda)^m a \\ 0 & 1 \end{pmatrix}.$$

These boundary conditions for (9) were earlier found in [2].

(vii) Equation (10):

$$L = \begin{pmatrix} \lambda + \frac{e^{q}}{e^{q}-e^{q_{-1,0}}} & (\lambda(1-\mu)-\mu)\frac{e^{q_{-1,0}}}{e^{q}-e^{q_{-1,0}}} \\ \frac{e^{q_{-1,0}}}{e^{q}-e^{q_{-1,0}}} & \lambda + 1 + (\lambda(1-\mu)-\mu)\frac{e^{q_{-1,0}}}{e^{q}-e^{q_{-1,0}}} \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda + \frac{e^{q}}{e^{q}-e^{q_{0,-1}}} & (\lambda(1-\mu)-\mu)\frac{e^{q_{0,-1}}}{e^{q}-e^{q_{0,-1}}} \\ \frac{e^{q_{0,-1}}}{e^{q}-e^{q_{0,-1}}} & \lambda + 1 + (\lambda(1-\mu)-\mu)\frac{e^{q_{0,-1}}}{e^{q}-e^{q_{0,-1}}} \end{pmatrix}$$

$$e^{q_{m,0}} = ce^{q_{m,1}} \qquad h(\lambda) = \lambda$$

$$H(m,\lambda) = \begin{pmatrix} \delta\frac{\beta}{\tau} + \left(\frac{c}{1-c}\right)^2\frac{\alpha}{\tau} + \delta\frac{c}{c-1}\frac{1}{\tau}g(m-1) & g(m) \\ -2\frac{c}{c-1}\frac{\beta}{\tau} - \left(\frac{c}{c-1}\right)^2\frac{1}{\tau}g(m-1) & -\left(\frac{c}{c-1}\right)^2\frac{\alpha}{\tau} + \gamma - \frac{c}{c-1}\frac{\gamma}{\beta}g(m-1) \end{pmatrix}$$

where

$$\alpha = \lambda - \lambda\mu - \mu \qquad \delta = \lambda + \frac{1}{c-1} \qquad \beta = 1 + \lambda\mu + \mu + \frac{\alpha}{c-1}$$

$$\tau = \left(\mu + \frac{1}{c-1} - \frac{1}{c-1}\mu\right)(\lambda+1)^2 \qquad \gamma = 1 + \frac{\alpha}{\tau}\left(1 - \frac{1}{c-1}\right)$$

$$c_1 = \left(\lambda + \frac{1}{c-1}\right)\frac{\gamma}{\beta} \qquad g(m) = 2\frac{\alpha\gamma c(1-c_1^m)}{\beta(c-1)(1-c_1)} + 2\beta(1-c)c_1^m\frac{1}{c}.$$

3. Finite-dimensional reductions and integrals of motion

It is natural to expect boundary conditions consistent with L, A pairs to preserve the integrability property of the equation. Unfortunately, the problem has not been investigated in detail, but below we will show that finite-dimensional reductions of equation (1) obtained by imposing boundary conditions consistent with the L, A pairs inherit a large number of integrals of motion. In order to construct the integrals of motion we make use of the idea proposed in [5].

Theorem. *Suppose that equation (1) is reduced to a finite interval with ends* $n_- = 1$ *and* $n_+ = N$ *by imposing boundary conditions* $q_{m,0} = f_-$ *and* $q_{m,N+1} = f_+$ *consistent with* L, A

pairs such that the corresponding H-matrices are $H_- = H_-(m,\lambda)$ and $H_+ = H_+(m,\lambda)$ respectively and involutions $h_-(\lambda)$ and $h_+(\lambda)$ coincide with each other. Then the function

$$g(\lambda) = \operatorname{tr}\left(T(m,\lambda)H_-^{-1}(m,\lambda)T^{-1}(m,h(\lambda))H_+(m,\lambda)\right)$$

where $T(m,\lambda) = L(m,N,\lambda)\cdots L(m,1,\lambda)$ is a generating function of the integrals of motion for the restricted chain, and tr A *means the trace of the matrix A.*

Proof. At the left end we have the equation

$$A(m,1,h(\lambda)) = H_-(m+1,\lambda)A(m,1,\lambda)H_-^{-1}(m,\lambda). \tag{24}$$

A similar equation holds at the right end:

$$A(m,N+1,h(\lambda)) = H_+(m+1,\lambda)A(m,N+1,\lambda)H_+^{-1}(m,\lambda). \tag{25}$$

From the zero-curvature representation

$$L(m+1,n)A(m,n) = A(m,n+1)L(m,n)$$

it follows that

$$A(m,1,\lambda) = T^{-1}(m+1,\lambda)A(m,N+1,\lambda)T(m,\lambda). \tag{26}$$

Replacing $\lambda \to h(\lambda)$ in (26) and simplifying by use of (24) and (25) leads to

$$\begin{aligned} A(m,1,\lambda) = {} & H_-^{-1}(m+1,\lambda)T^{-1}(m+1,h(\lambda))H_+(m+1,\lambda)A(m,N+1,\lambda) \\ & \times H_+^{-1}(m,\lambda)T(m,h(\lambda))H_-(m,\lambda). \end{aligned} \tag{27}$$

Equating the right-hand sides of (26) and (27) gives

$$A(m,N+1,\lambda) = P(m+1,\lambda)A(m,N+1,\lambda)P^{-1}(m,\lambda)$$

where $P(m,\lambda) = T(m,\lambda)H_-^{-1}(m,\lambda)T^{-1}(m,h(\lambda))H_+(m,\lambda)$. Now it is easy to see that $\operatorname{tr} P(m+1,\lambda) = \operatorname{tr} P(m,\lambda)$, i.e. the trace does not depend on time m. The theorem is proved. □

Example 1. Consider the Heisenberg equation (3) restricted to a finite interval by imposing the boundary conditions $q_{m,0} = \infty$ and $q_{m,N+1} = \infty$. This finite-dimensional chain has N functionally independent integrals of motion:

$$I_1 = \sum_{i=1}^{N} \frac{1}{q_{m,i} - q_{m-1,i}}$$

$$I_2 = \sum_{i=2}^{N} \frac{1}{q_{m,i} - q_{m-1,i}} \sum_{j=1}^{i-1} \frac{q_{m,i} - q_{m-1,j}}{q_{m,j} - q_{m-1,j}}$$

$$I_3 = \sum_{i=3}^{N} \frac{1}{q_{m,i} - q_{m-1,i}} \sum_{j=2}^{i-1} \frac{q_{m,i} - q_{m-1,j}}{q_{m,j} - q_{m-1,j}} \sum_{k=1}^{j-1} \frac{q_{m,j} - q_{m-1,k}}{q_{m,k} - q_{m-1,k}}$$

$$\cdots$$

$$I_n = \sum_{j_1=n}^{N} \frac{1}{q_{m,j_1} - q_{m-1,j_1}} \prod_{i=1}^{n-1} \sum_{j_i=i}^{j_{i+1}-1} \frac{q_{m,j_{i+1}} - q_{m-1,j_i}}{q_{m,j_i} - q_{m-1,j_i}}.$$

Example 2. In the case of Toda chain (4) with the boundary conditions $q_{m,0} = \infty$ and $q_{m,N+1} = -\infty$ the following integrals can easily be found:

$$I_1 = \sum_{i=1}^{N} e^{q_{m,i} - q_{m-1,i}}$$

$$I_2 = -e^{q_{m,N} - q_{m-1,N-1}} + \sum_{i=2}^{N} e^{q_{m,i} - q_{m-1,i}} \sum_{j=1}^{i-1} e^{q_{m,j} - q_{m-1,j}}$$

$$\begin{aligned} I_3 = -e^{q_{m,N} - q_{m-1,N-1}} &(e^{q_{m,N-2} - q_{m-1,N-2}} + e^{q_{m,N-3} - q_{m-1,N-3}}) \\ &+ e^{q_{m,N} - q_{m-1,N}} e^{q_{m,N-1} - q_{m-1,N-2}} + e^{q_{m,N} - q_{m,N-2}} \\ &+ \sum_{i=3}^{N} e^{q_{m,i} - q_{m-1,i}} \sum_{j=2}^{i-1} e^{q_{m,j} - q_{m-1,j}} \sum_{k=1}^{j-1} e^{q_{m,k} - q_{m-1,k}}. \end{aligned}$$

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